Efficient computation of topological entropy, pressure, conformal measures, and equilibrium states in one dimension

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We describe a fast and accurate method to compute the pressure and equilibrium states for maps of the interval $T:[0,1]\to[0,1]$ with respect to potentials $\phi:[0,1]\to\mathbb{R}$. An approximate Ruelle-Perron-Frobenius operator is constructed and the pressure read off as the logarithm of the leading eigenvalue of this operator. By setting $\phi\equiv 0$, we recover the topological entropy. The conformal measure and the equilibrium state are computed as eigenvectors. Our approach is extremely efficient and very simple to implement. Rigorous convergence results are stated for piecewise expanding maps.

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I. BACKGROUND AND INTRODUCTION

Given a piecewise C^1 interval map $T:[0,1]\to[0,1]$, the Perron-Frobenius operator $\mathcal{P}:L^1([0,1])\to L^1([0,1])$, $\mathcal{P}f(x):=\sum_{y\in T^{-1}x}\frac{f(y)}{|T'(y)|}$ describes the action of T on ensembles of initial conditions defined by densities $f:[0,1]\to\mathbb{R}^+$. If a density f is fixed by $\mathcal{P}$, then f is a T-invariant density and for ergodic T characterizes the long term distribution of orbits starting in the support of f.

A well known generalization of this operator is the Ruelle-Perron-Frobenius (RPF) operator [1,2] $\mathcal{L}_\phi:\mathcal{R}\to\mathcal{R}$, $\mathcal{L}_\phi f(x):=\sum_{y\in T^{-1}x}e^{\phi(y)}f(y)$ where $\mathcal{R}$ is a space of real or complex valued functions on $[0,1]$ with some form of regularity. In suitable settings, thermodynamic quantities such as *pressure* and *equilibrium states* can be read off from an RPF operator. Nonanalyticity of the pressure and associated bifurcations of the equilibrium states are associated with *phase transitions*, and these features of RPF operators have been the subject of intense research since the 1980s [2–7] (see also [1] and the references contained therein). Recent progress by Sarig [8], Pesin and Zhang [9,10], Yuri [11], and others has advanced the rigorous study of the statistical properties of certain systems at points of phase transition (non-Gaussian limit behaviors of ergodic averages, subexponential mixing rates, and so on). Our present study complements these developments by offering a simple, fast, and accurate numerical tool for the analysis of RPF operators and their associated thermodynamical objects.

A probability measure μ_ϕ is called an equilibrium state (ES) [12] for the pair (T,ϕ) if it realizes the following maximum: $P(\phi)=\sup\{h_\mu+\int\phi d\mu:\mu=\mu\circ T^{-1},\mu([0,1])=1\}=h_{\mu_\phi}+\int\phi d\mu_\phi$, where h_μ is the metric entropy of T and $P(\phi)$ is the pressure.

If T is piecewise monotonic and forward transitive, and e^ϕ is finite, of bounded variation, and suitably contractive along orbits, then (see, for example, [1,13]) $\mathcal{L}_\phi$ has a simple largest positive eigenvalue $\lambda_\phi=\exp(P(\phi))$, and a unique ES $\mu_\phi=h_\phi\nu_\phi$, where

$$\mathcal{L}_\phi h_\phi=\lambda_\phi h_\phi, \tag{1}$$

$$\mathcal{L}_\phi^*\nu_\phi=\lambda_\phi\nu_\phi, \tag{2}$$

and the action of the dual operator $\mathcal{L}_\phi^*$ on a probability measure ν satisfies $(\mathcal{L}_\phi^*\nu)(f)=\nu[\mathcal{L}_\phi(f)]$ for all bounded variation $f:[0,1]\to\mathbb{R}$. The probability measure ν_ϕ is known as a *conformal measure*. The eigenfunction h_ϕ is the density of the ES μ_ϕ with respect to the conformal measure ν_ϕ. Thus μ_ϕ is absolutely continuous with respect to ν_ϕ.

We describe a new method for numerically calculating the topological pressure (with the topological entropy as a special case), the conformal measure, and the ES for a piecewise monotonic interval map $T:[0,1]\to[0,1]$ and a weight function $\exp(\phi)$. Our approach has its origins in "Ulam's method" [14] which has been used successfully for the approximation of the physical invariant measures.

Several authors have treated the numerical approximation of the topological entropy [15–19] and pressure [4] of interval maps via transfer operator or matrix techniques. There are many more papers dedicated to calculations for specific maps, particularly those that display intermittency (e.g., [5–7]). The approaches [4,15,18,19] require the use of recursive calculations to keep track of multiple inverse branches or to approximate T by a Markov map. Notable among these is the paper of Kovács and Tel [20] where a variant of Ulam's method is used to estimate the eigenfunctions of certain RPF operators. Their approach depends explicitly on the recursive calculation of a Markov partition. Our approach does not depend on such considerations; rather (i) single iterates of T and (ii) evaluations of ϕ are carried out at many sample points and directly assembled into a matrix. For a relatively general class of piecewise monotonic maps and potentials, our method produces estimates of topological entropy and pressure that converge to the exact values [21]. Furthermore, we can rigorously approximate the density h_ϕ in the L^1 norm. In this paper we demonstrate numerically that our method is extremely effective at estimating thermodynamical quantities for very general maps including those that are nonhyperbolic or display intermittent behavior. The key benefits of our approach are (i) the numerical implementation is simple, (ii) it does not rely on any special structural properties of the dynamics, and (iii) rigorous convergence results continue to be developed.

II. DEVELOPMENT OF THE NUMERICAL METHOD

Partition $[0,1]$ into n subintervals $A_1,\ldots,A_n$ and define the subspace $\Delta_n=\mathrm{sp}\{\chi_{A_1},\ldots,\chi_{A_n}\}$, and the canonical projection $\pi_n:L^1([0,1])\to\Delta_n$, $\pi_n f:=\sum_{i=1}^n \frac{1}{m(A_i)}\int_{A_i} f\,dm$, where m denotes Lebesgue measure on $[0,1]$. We consider the action of $\mathcal{L}_\phi$ on Δ_n by defining a projected operator $\mathcal{L}_{n,\phi}:=\pi_n\mathcal{L}_\phi$. This action is described by a matrix equation [21]:

$$\pi_n\mathcal{L}_\phi\left(\sum_{i=1}^n a_i\chi_{A_i}\right)=\sum_{j=1}^n\left(\sum_{i=1}^n a_i L_{n,\phi,ij}\right)\chi_{A_j}, \tag{3}$$

where $L_{n,\phi,ij}=\frac{1}{m(A_j)}\int_{A_i\cap T^{-1}A_j} e^{\phi(y)}|T'(y)|dy$. In practice, we use an equipartition, setting $A_i:=\left[\frac{i-1}{n},\frac{i}{n}\right)$, $i=1,\ldots,n$. The entries of $L_{n,\phi,ij}$ may be computed using some form of numerical integration. In the one-dimensional experiments reported here, we select a uniform sample of N points $x_{ij,1},\ldots,x_{ij,N}\in A_i\cap T^{-1}A_j$ and approximate each integral $\int_{A_i\cap T^{-1}A_j} e^{\phi(y)}|T'(y)|dy$ by $\frac{m(A_i\cap T^{-1}A_j)}{N}\sum_{k=1}^N e^{\phi(x_{ij,k})}|T'(x_{ij,k})|$. Typically we have used $100\le N\le 1000$. A simpler method of estimating L, especially useful in cases where there is no easily defined inverse map, is to sample N points $x_{i,1},\ldots,x_{i,N}$ uniformly distributed in each set A_i, $i=1,\ldots,n$ and use

$$L_{n,\phi,ij}\approx\frac{\sum_{k:x_{i,k}\in A_i,T(x_{i,k})\in A_j} e^{\phi(x_{i,k})}|T'(x_{i,k})|}{N}. \tag{4}$$

If T is not uniformly expanding, one should take care when applying Eq. (4) to use a sufficiently large value for N relative to the number of partition sets n.

Let $\lambda_{n,\phi}$ denote the leading eigenvalue of $L_{n,\phi}$ and $\ell_{n,\phi}$, $r_{n,\phi}$ be the corresponding left and right eigenvectors; that is, $\ell_{n,\phi}L_{n,\phi}=\lambda_{n,\phi}\ell_{n,\phi}$ and $L_{n,\phi}r_{n,\phi}=\lambda_{n,\phi}r_{n,\phi}$. In the following sections, we discuss the approximation of thermodynamical quantities in the $n\to\infty$ limit, where it is understood that $\max_{1\le i\le n}m(A_i)\to 0$ as $n\to\infty$.

A. Topological entropy and pressure

In the sequel we consider the well-studied family of potentials $\phi=-\beta\log|T'|$, $\beta\in\mathbb{R}$. Throughout the paper log denotes the natural logarithm $\log_e$. To estimate the pressure using our method we compute the leading eigenvalue of the matrix

$$L_{n,\phi,ij}=\frac{1}{m(A_j)}\int_{A_i\cap T^{-1}A_j}|T'(y)|^{1-\beta}dy. \tag{5}$$

If T is expanding, piecewise C^2, and covering, Terhesiu and Froyland [21] prove that $P_{n,\phi}:=\log\lambda_{n,\phi}\to P(\phi)$ as $n\to\infty$. In Sec. III we demonstrate that our approach is very accurate for more general T. Note that when $\beta=1$, Eq. (5) simplifies to the standard Ulam matrix approximating the Perron-Frobenius operator, namely, $L_{n,1,ij}=m(A_i\cap T^{-1}A_j)/m(A_j)$. In the case where $\beta=0$, the topological entropy is estimated by $\log(\lambda_{n,0})$ where $\lambda_{n,0}$ is the leading eigenvalue of

$$L_{n,0,ij}=\frac{1}{m(A_j)}\int_{A_i\cap T^{-1}A_j}|T'(y)|dy. \tag{6}$$

This turns out to be efficient, easy, and accurate, the latter due in part to the use of derivative information.

B. Approximating the density of the ES, the conformal measure, and the ES

Define $h_{n,\phi}=\sum_{i=1}^n \ell_{n,\phi,i}\chi_{A_i}$. Then $h_{n,\phi}$ satisfies $\pi_n\mathcal{L}_\phi h_{n,\phi}=\lambda_{n,\phi}h_{n,\phi}$, and we expect that $h_{n,\phi}\to h_\phi$ as $n\to\infty$. Terhesiu and Froyland [21] prove that $\|h_{n,\phi}-h_\phi\|_{L^1}\to 0$ as $n\to\infty$ in the setting of Sec. II A and numerical experiments demonstrate remarkable accuracy for more general maps.

For the approximation of the conformal measure, we define a probability measure $\nu_{n,\phi}$ by $\nu_{n,\phi}(A):=\sum_{i=1}^n\frac{m(A\cap A_i)}{m(A_i)}r_{n,\phi,i}$, where $r_{n,\phi,i}$ has been suitably normalized. We outline an argument that supports our contention that $\nu_{n,\phi}$ is a good approximation of the conformal measure ν_ϕ. Consider $[(\pi_n\mathcal{L}_\phi^*)\nu](A_i)=\nu(\pi_n\mathcal{L}_\phi\chi_{A_i})=\nu(\sum_{j=1}^n L_{n,\phi,ij}\chi_{A_i})$ $=\sum_{j=1}^n L_{n,\phi,ij}\nu(A_j)$. Thus $[(\pi_n\mathcal{L}_\phi^*)\nu_{n,\phi}](A_i)=\sum_{j=1}^n L_{n,\phi,ij}\nu_{n,\phi}(A_j)$ $=\lambda_{n,\phi}\nu_{n,\phi}(A_i)$ and so for the sets $A_1,\ldots,A_n$, $\nu_{n,\phi}$ satisfies the appropriate eigenvalue equation (2). We demonstrate via numerical experiments in Sec. III that $\nu_{n,\phi}$ well approximates ν_ϕ; a proof of this fact in the setting of Sec. II A is work in progress.

Finally, for the approximation of the ES, we define a probability measure $\mu_{n,\phi}$ by $\mu_{n,\phi}(A):=\sum_{i=1}^n\frac{m(A\cap A_i)}{m(A_i)}\ell_{n,\phi,i}r_{n,\phi,i}$, where $\ell_{n,\phi,i}$ and $r_{n,\phi,i}$ have been suitably normalized. Based on the above arguments we contend that $\mu_{n,\phi}\to\mu_\phi$ as $n\to\infty$ and again demonstrate a high degree of accuracy in Sec. III.

III. NUMERICAL EXPERIMENTS

It is well known that nonhyperbolic dynamical systems can provide good models for phase transitions from a periodic state to a chaotic one as the temperature is varied. Phase transitions are described by nonanalytic behavior of the pressure function $\beta\mapsto P(-\beta\log|T'|)$ (see, for instance, [22,23,4,6,7]). We consider two examples: the logistic family, whose dynamics are nonexpanding, and the Farey map, whose indifferent fixed point at 0 leads to intermittency and a phase transition.

A. Example 1: Logistic family

$$T(x)=rx(1-x),\quad r\in[0,4].$$

As a first test of our method we set $\beta=0$ and calculated the topological entropy h_{top} at a range of parameter values. The results are depicted in Fig. 1(a). For comparison, we implemented the Block *et al.* algorithm [16] to compute h_{top} to an accuracy of 10^{-4}; the agreement with the results depicted in Fig. 1(a) is very good; the error is displayed in Fig. 1(b). Note that for values of $r>3.6$ our generalized Ulam method is accurate to within 10^{-3} once $n=10^4$. We remark that the algorithm in [16] exploits the kneading invariant—a rela-

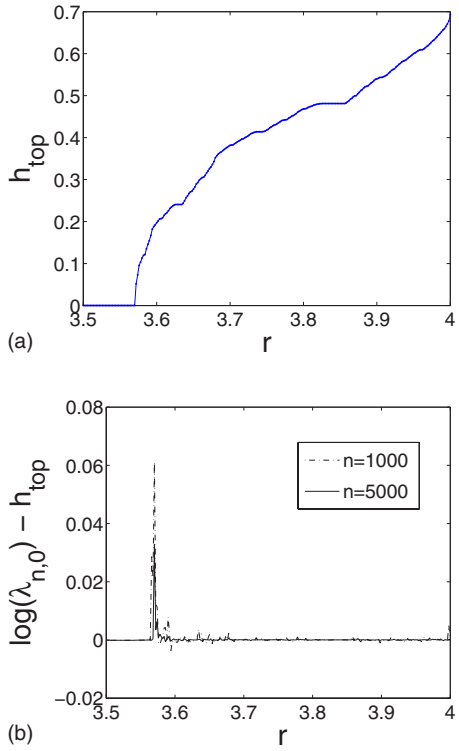

FIG. 1. (Color online) (a) Topological entropy for the logistic map computed as described in Sec. II A, $n=5000$ bins, $N=100$ test points. (b) Discrepancies in entropy calculations for the logistic map ($n=1000$ and $n=5000$); $\lambda_{n,0}$ is leading eigenvalue of $L_{n,0}$, h_{top} computed to 10^{-4} using the Block *et al.* algorithm [16].

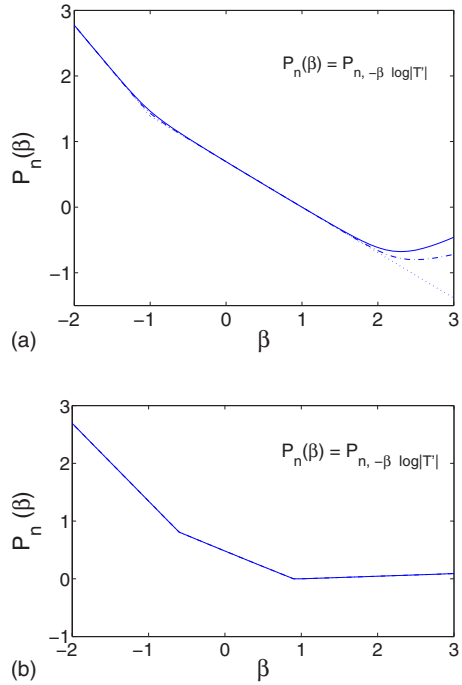

FIG. 2. (Color online) (a) Pressure $P_{n,-\beta\log|T'|}$ computed by Ulam's method for logistic map with $r=4$, $n=1000$ (solid), and $n=10000$ (dot-dash); exact result displayed as a dotted line. (b) Pressure $P_{n,-\beta\log|T'|}$ computed by Ulam's method for logistic map with $r=3.84$, $n=1000$ (dot-dash), and $n=10000$ (solid) indistinguishable.

tively sophisticated construction which exploits special properties of unimodal maps. By contrast, our method uses only "one-step" dynamical information about the evolution of individual points, and knowledge of the derivative of the map.

Figure 2(a) depicts the results of a pressure computation for the Ulam–von Neumann logistic map (parameter $r=4$). For $\beta<-1$, the ES at this parameter value is a δ measure concentrated on $x=0$, and for $\beta>-1$ is the absolutely continuous invariant measure (ACIM) with density $1/[\pi\sqrt{x(1-x)}]$, which is also a measure of maximal entropy. Thus [3]

$$P(-\beta\log|T'|)=\begin{cases}-2\beta\log 2 & \beta<-1,\\ (1-\beta)\log 2 & \beta>-1\end{cases}$$

with a phase transition occurring at $\beta=-1$. This switching of the ES is revealed by our calculations; the phase transition is obvious in the plots in Fig. 2(a).

Convergence for $\beta>1$ is slow, however, and we are presently seeking an explanation. Numerical tests at sample β values ($\beta=-0.5$, 0, and 0.5) confirm that our method returns estimates $h_{n,\phi}$ and $\nu_{n,\phi}$ that are very close to the theoretical objects. Since the parameter $r=4.0$ is rather special from a thermodynamic viewpoint (the ACIM is in fact a measure of maximal entropy), we also depict in Fig. 2(b) the results of a pressure calculation for the parameter value $r=3.84$, where the logistic map has a stable period-3 orbit. Here, we observe three interesting phases. First of all, for sufficiently negative β the δ measure on the unstable fixed point at $x=0$ is an ES, and the topological pressure is $P(-\beta\log|T'|)=-\beta\log 3.84$. For $\beta>1$ the stable period-3 orbit at approximately $\{0.1494, 0.4880, 0.9594\}$ (with Lyapunov exponent $\lambda_{per}\approx -0.0444$) determines the pressure: the ES is a sum of δ measures on the period–3 orbit, and $P(-\beta\log|T'|)=-\beta\lambda_{per}$ [using [24], Theorem 4.1, and convexity of $P(\cdot)$]. This phase is also clearly visible in Fig. 2(b). In between these extremes the pressure is determined by the dynamics of T on a repelling Cantor set.

B. Example 2: Farey map

$$T(x)=\begin{cases}\frac{x}{1-x} & \text{if } x\le 1/2,\\ \frac{1-x}{x} & \text{if } x>1/2.\end{cases}$$

Sarig [23] gives a convenient and general analysis of the thermodynamic formalism for systems that can be modeled by the renewal shift. By applying Sarig's results to the Farey map one obtains that there is a critical β_c for which (i) when $\beta<\beta_c$, $P(-\beta\log|T'|)$ is analytic and there are unique finite ESs; (ii) $P(-\beta\log|T'|)$ is nonanalytic at $\beta=\beta_c$ and supports a continuous (but σ-finite) invariant measure as well as a singular equilibrium measure; (iii) for $\beta>\beta_c$, $P(-\beta\log|T'|)$ is linear, and there are no continuous equilibrium measures [hence no continuous solutions to Eq. (2)]. Prellberg [6] showed that the leading eigenvalue $\lambda(\beta)$ of the RPF operator $\mathcal{L}_\phi$ associated with T is real analytic for all $\beta<1$ and all $\beta>1$ with a nonanalyticity at $\beta=1$. He further showed that $\log[\lambda(\beta)]$ can be identified with $P(-\beta\log|T'|)$ and that the pressure function $\beta\mapsto P(-\beta\log|T'|)$ therefore

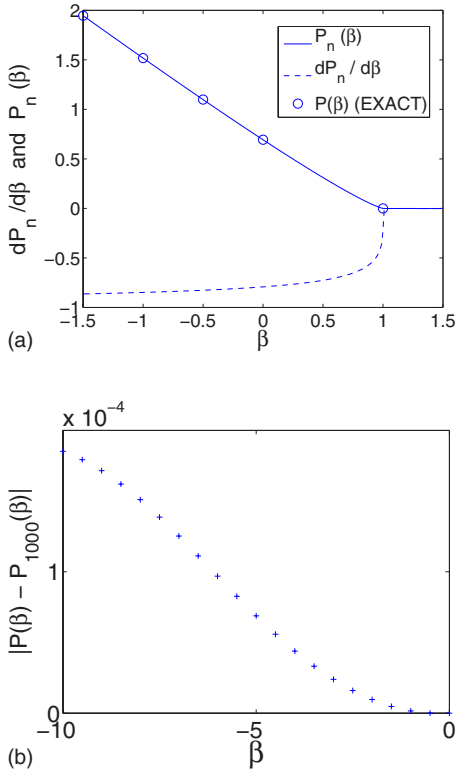

FIG. 3. (Color online) (a) Pressure approximations $P_{n,-\beta\log|T'|} =: P_n(\beta)$ (solid) and $dP_n/d\beta$ (dashed) for the Farey map computed via Ulam's method ($n=1000$ bins, $N=100$ test points per partition element). Several exact values are circled. (b) Error between exact pressure values and Ulam approximations when $\beta=-m/2$ (see text).

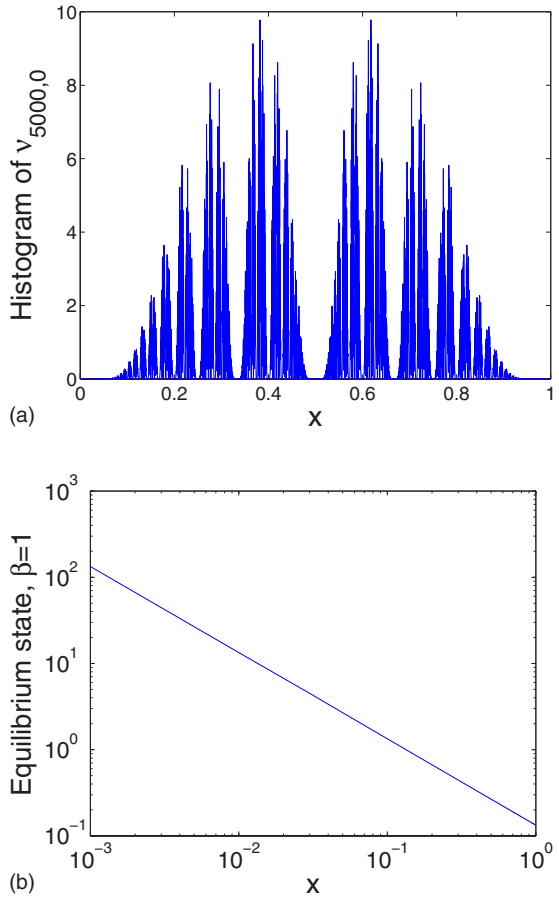

FIG. 4. (Color online) (a) Ulam approximate conformal measure ($n=5000$) for the Farey map with $\beta=0$. (b) log-log plot of Ulam approximation to the Farey map ES ($n=1000$), $\beta=1$.

encounters a nonanalyticity (corresponding to a phase transition) at $\beta=1$. Thus $\beta_c=1$ and the thermodynamic phases of the Farey map can be studied by transfer operator methods. Our generalized Ulam method confirms these predictions with a very high precision: Figure 3(a) shows the numerically computed pressure function $P(-\beta\log|T'|)$ and its derivative; the phase transition at $\beta=1$ is clearly visible, showing up as a "kink" in the graph of P. We summarize our findings in the three temperature regimes as follows.

$\beta<1$. The ES is a continuous probability measure [6,23] and $P(-\beta\log|T'|)$ is real analytic. In fact, when $\beta=-m/2$ ($0\leq m\in\mathbb{Z}$), the recurrence relation (1) has polynomial solutions [4], and the value of $P[(m/2)\log|T'|]$ can be exactly calculated as the logarithm of the leading eigenvalue of an $(m+1)\times(m+1)$ matrix. Figures 3(a) and 3(b) show excellent agreement between these values and the calculations based on our extended Ulam's method ($n=1000$, $N=100$); our estimates are accurate to within 2×10^{-4} for $m=0,1,\ldots,20$. As discussed in Sec. II B, our method also provides approximate conformal measures and ESs for arbitrary β. Figure 4(a) displays the results of a sample calculation with $\beta=0$: the measure is represented as a histogram on $n=5000$ equally spaced bins. We have chosen $\beta=0$ as the ES at $\beta=0$ is a measure of maximal entropy, and the density $h_0\equiv1$; thus $\mu_0=\nu_0$. Furthermore, by exploiting the conjugacy with the two-shift we can obtain independent numerical estimates of the ES μ_0 against which we can compare our Ulam estimate. The agreement is excellent: using the Hutchinson metric, the weak-* discrepancy between μ_0 and $\mu_{5000,0}$ (both represented as 5000 bin histograms) was $1.07\times10^{-5}\pm2\times10^{-4}$. The dominant error of 2×10^{-4} is due to the representation as a 5000 bin histogram; the error in mass assigned to each bin is extremely small.

$\beta=1$. At $\beta=1$ there co-exist a σ-finite invariant measure μ (with $\frac{d\mu}{dx}=\frac{1}{x}$) and δ_0—the point mass on the indifferent fixed point at 0. Interestingly, Ulam's method detects μ rather than δ_0. In Fig. 4(b) we depict a log-log plot of the Ulam approximate ES ($n=1000$); a slope of almost exactly -1 indicates a density consistent with $1/x$. We believe that in this case δ_0 is not selected by the extended Ulam's method because of the instability of δ_0 to stochastic perturbations; the fixed point at $x=0$ is weakly unstable and once perturbed from 0, orbits of the dynamical system are distributed like $1/x$.

$\beta>1$. In this regime, the continuous invariant measures are no longer ESs [23], and $P(-\beta\log|T'|)=0$. Numerical experiments with $\beta=1.1$ suggest that the "approximate ES" $\mu_{n,\phi}$ approaches δ_0 as $n\to\infty$.

IV. DISCUSSION AND CONCLUSIONS

We have described a simple, fast, and accurate numerical method for estimating important thermodynamic quantities

associated with one-dimensional dynamical systems. Using the logistic and Farey maps as case studies we have shown that the numerical method is a straightforward and accurate way to approximate the pressure, detect phase transitions, and estimate ESs. Rigorous convergence results for the pressure and densities of ESs exist for a broad class of expanding maps [21]. Proofs of the convergence of the approximate ESs $\mu_{n,\phi}$ to μ_ϕ and extensions to nonuniformly expanding maps are currently being investigated.

[1] V. Baladi, *Positive Transfer Operators and Decay of Correlations* (World Scientific, Singapore, 2000).
[2] D. Ruelle, Commun. Math. Phys. **125**, 239 (1989).
[3] T. Bohr and D. Rand, Physica D **25**, 387 (1987).
[4] J. M. Feigenbaum, I. Proccaccia, and T. Tel, Phys. Rev. A **39**, 5359 (1989).
[5] Z. Kaufmann, H. Lustfeld, and J. Bene, Phys. Rev. E **53**, 1416 (1996).
[6] T. Prellberg, Ph.D. thesis, Virginia Polytechnic Institute, 1991.
[7] X.-J. Wang, Phys. Rev. A **39**, 3214 (1989).
[8] O. Sarig, Commun. Math. Phys. **267**, 631 (2006).
[9] Y. Pesin and K. Zhang, J. Stat. Phys. **122**, 1095 (2006).
[10] Y. Pesin and K. Zhang, *Thermodynamics Associated with Inducing Schemes and Liftability of Measures*, Proc. Fields Inst. (in press).
[11] M. Yuri, Trans. Am. Math. Soc. **355**, 2949 (2003).
[12] D. Ruelle, *Thermodynamic Formalism*, Second edition (Cambridge University Press, Cambridge, England, 2004).
[13] G. Keller, *Equilibrium States in Ergodic Theory*, LMS student texts Vol. 42 (Cambridge University Press, Cambridge, England, 1998).
[14] T.-Y. Li, J. Approx. Theory **17**, 177 (1976).
[15] N. J Balmforth, E. A. Spiegel, and C. Tresser, Phys. Rev. Lett. **72**, 80 (1994).
[16] L. Block, J. Keesling, S. Li, and K. Peterson, J. Stat. Phys. **55**, 929 (1989).
[17] P. Collet, J. P. Crutchfield, and J.-P. Eckmann, Commun. Math. Phys. **88**, 257 (1983).
[18] P. Gora and A. Boyarsky, Trans. Am. Math. Soc. **323**, 39 (1991).
[19] T. Steinberger, J. Stat. Phys. **95**, 287 (1999).
[20] Z. Kovács and T. Tel, Phys. Rev. A **40**, 4641 (1989).
[21] D. Terhesiu and G. Froyland (unpublished).
[22] C. Maes, F. Redig, F. Takens, A. van Moffaert, and E. Verbitski, Nonlinearity **13**, 1681 (2000).
[23] O. Sarig, Commun. Math. Phys. **217**, 555 (2001).
[24] H. Bruin and G. Keller, Ergod. Theory Dyn. Syst. **18**, 765 (1998).